QUIZ 7: LESSON 8
SEPTEMBER 14, 2018

Write legibly, clearly indicate the question you are answering, and put a box or circle around your final answer. If you do not clearly indicate the question numbers, I will take off points. Write as much work as you need to demonstrate to me that you understand the concepts involved. If you have any questions, raise your hand and I will come over to you.

A 500-gallon tank initially contains 300 gallons of pure distilled water. Brine containing 4 lbs of salt per gallon flows into the tank at a rate of 5 gallons per minute and the well-stirred mixture flows out of the tank at a rate of 5 gallons per minute.

1. [4 pts] Set up a differential equation describing the change in the amount of salt in the tank at time t.

Solution: Let $A(t)$ be the pounds of salt in the tank at time t minutes. We need to find an equation describing $\frac{dA}{dt}$. Note that $\frac{dA}{dt}$ is measured in pounds per minute. The change in the amount of salt in the tank $\left(\frac{dA}{dt}\right)$ is the difference between the amount of salt entering the tank and the amount of salt leaving the tank. So we break this into two pieces:

[Salt in]: Each minute, 5 gallons flow into the tank and, in each gallon, there are 4 pounds of salt. Hence,

$$[\text{Salt in}] = \underbrace{\left(\frac{5 \text{ gal}}{1 \text{ min}}\right)}_{\text{gal/min}} \underbrace{\left(\frac{4 \text{ lbs}}{1 \text{ gal}}\right)}_{\text{lbs/gal}} = 20 \text{ lbs/min}$$

[Salt out]: Well-stirred in this case means that each gallon of liquid in the tank has as much salt as any other gallon. So, we take the total amount of salt and divide by the total number of gallons of liquid in the tank. However, observe that as water is pouring in and out, the amount of liquid in the tank *may* vary per minute.

By definition, $A(t)$ is the total amount of salt in the tank each minute. Further, we start with 300 gallons of liquid and, each minute, 5 gallons flow into the tank while 5 gallons flow out of the tank. Hence,

$$\text{Number of gallons in tank} = 300 + \underset{\substack{\uparrow \\ \text{gal in} \\ \text{per min}}}{5t} - \underset{\substack{\uparrow \\ \text{gal out} \\ \text{per min}}}{5t} = 300.$$

In this case, we see the number of gallons in the tank is constant, but this is not always the case. From this we have

$$[\text{Salt out}] = \underbrace{\left(\frac{5\text{ gal}}{1\text{ min}}\right)}_{\text{gal/min}}\underbrace{\left(\frac{A}{300}\right)}_{\text{lbs/gal}} = \frac{5A}{300} = \frac{A}{60}.$$

Since $\dfrac{dA}{dt} = [\text{Salt in}] - [\text{Salt out}]$, we have

$$\boxed{\frac{dA}{dt} = 20 - \frac{A}{60}}.$$

2. [5 pts] Find the particular solution to the differential equation in # 1.

<u>Solution</u>: The differential equation

$$\frac{dA}{dt} = 20 - \frac{A}{60} = \frac{1200}{60} - \frac{A}{60} = \frac{1200 - A}{60}$$

is separable. We write

$$\begin{aligned} \frac{dA}{dt} &= \frac{1200 - A}{60} \\ \Rightarrow \quad \frac{60}{1200 - A}\, dA &= dt \\ \Rightarrow \quad \int \frac{60}{1200 - A}\, dA &= \int dt \end{aligned}$$

The LHS is a u-sub problem.

Let $u = 1200 - A$, then $du = -dA$. Thus,

$$\begin{aligned} \int \frac{60}{1200 - A}\, dA &= 60 \int \frac{1}{u}\,(-du) \\ &= -60 \int \frac{1}{u}\, du \\ &= -60 \ln u \\ &= -60 \ln(1200 - A) \end{aligned}$$

So we write

$$\begin{aligned} \int \frac{60}{1200 - A}\, dA &= \int dt \\ \Rightarrow \quad -60 \ln(1200 - A) &= t + C \\ \Rightarrow \quad \ln(1200 - A) &= -\frac{t}{60} + C \\ \Rightarrow \quad 1200 - A &= e^{-t/60 + C} = Ce^{-t/60} \\ \Rightarrow \quad A &= 1200 - Ce^{-t/60} \end{aligned}$$

Because we originally begin with 300 gallons of pure distilled water, there is no salt in the tank and so $A(0) = 0$. This means that

$$\underbrace{0}_{A(0)} = 1200 - C\underbrace{e^0}_{1} = 1200 - C$$

and so $C = 1200$. Thus

$$\boxed{A = 1200 - 1200e^{-t/60}}$$

3. [1 pt] Find the amount of salt in the tank after 10 minutes. Round your answer to 2 decimal places.

<u>Solution</u>: By # 2, we write

$$A(10) = 1200 - 1200e^{-10/60} \approx \boxed{184.22 \text{ lbs}}.$$